The Green-Kubo formula for locally interacting open fermionic systems

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Abstract

We consider a model describing finitely many free Fermi gas reservoirs coupled by local interactions and prove the Green-Kubo formulas and the Onsager reciprocity relations for heat and charge fluxes generated by temperature and chemical potential differentials.

1 Introduction

This is the fourth in a series of papers [JOP1, JOP2, JOP3] dealing with derivation of Green-Kubo formulas (GKF) and Onsager reciprocity relations (ORR) in quantum statistical mechanics. The first two papers [JOP1, JOP2] were devoted to the abstract axiomatic derivation of GKF and ORR for open systems driven by thermodynamical forces associated to temperature and chemical potential differentials. This paper and [JOP3] are devoted to the study of concrete models.

In [JOP3] we have studied the well-known spin-fermion model describing the interaction of an N-level atom with finitely many independent free Fermi gas reservoirs [Da, LeSp, JP2]. Combining the results of [JOP1, JOP2] with spectral theory of non-equilibrium steady states developed in [JP2] we have established GKF and ORR for this class of models.

In this paper we study a model describing finitely many free Fermi gas reservoirs coupled by local interactions and show that the abstract derivation of [JOP1, JOP2] combined with scattering theory of non-equilibrium steady states (see [BM1, AM, BM2, Ru1, FMU]) yields the GKF and ORR for this class of models.

Throughout the paper we shall assume that the reader is familiar with general aspects of linear response theory discussed in [JOP1, JOP2, JOP3] and with the algebraic formalism of quantum statistical mechanics [BR1, BR2]. A modern introduction to these topics can be found in [JP3, FMU] and in the recent lecture notes [AJPP1].

The paper is organized as follows. In Subsection 1.1 for notational purposes we review the description of a free Fermi gas in the algebraic formalism of quantum statistical mechanics. In Subsection 1.2 we introduce the model and state our results. The strategy of the proof is the same as in [JOP3] and is described in Section 3.1. This strategy reduces the proof of all our results to a technical estimate formulated in Theorem 3.1. This estimate, which is our main technical result, is established in Section 3.2.

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1.1 Preliminaries

Let $\mathfrak{h}$ and h_0 be given Hilbert space and Hamiltonian. The corresponding free Fermi gas is described by the C^*-dynamical system $(\mathcal{O}, \tau_0)$ where:

(i) $\mathcal{O} = \mathrm{CAR}(\mathfrak{h})$ is the CAR algebra over $\mathfrak{h}$. We denote by $a^*(f)/a(f)$ the creation/annihilation operator associated to $f \in \mathfrak{h}$. As usual, $a^\#$ stands for either a or a^*;

(ii) τ_0^t is the group of Bogoliubov $*$-automorphisms generated by h_0, $\tau_0^t(a^\#(f)) = a^\#(\mathrm{e}^{\mathrm{i}th_0}f)$. We denote by δ_0 the generator of τ_0;

The gauge group of the free Fermi gas is the group of Bogoliubov $*$-automorphisms ϑ^φ, $\varphi \in \mathbb{R}$, generated by the identity operator on $\mathfrak{h}$. The physical observables are gauge invariant and hence elements of

$$\mathcal{O}_\vartheta = \{A \in \mathcal{O} \,|\, \vartheta^\varphi(A) = A \text{ for all } \varphi \in \mathbb{R}\}.$$

$\mathcal{O}_\vartheta$ is the τ_0-invariant C^*-subalgebra of $\mathcal{O}$ generated by $\{a^*(f)a(g) \,|\, f, g \in \mathfrak{h}\}$ and $\mathbb{1}$.

Let $\beta > 0$ and $\mu \in \mathbb{R}$ be parameters and $\omega_{\beta\mu}$ the gauge-invariant quasi-free state on $\mathcal{O}$ generated by

$$T_{\beta\mu} = \frac{1}{1 + \mathrm{e}^{\beta(h_0 - \mu)}}.$$

The quantum dynamical system $(\mathcal{O}, \tau_0, \omega_{\beta\mu})$ describes a free Fermi gas in thermal equilibrium at inverse temperature β and chemical potential μ. We remark that $\omega_{\beta\mu}$ is the unique β-KMS state for the C^*-dynamics $\tau_0^t \circ \vartheta^{-\mu t}$ and that $\omega_{\beta\mu} \restriction \mathcal{O}_\vartheta$ is a (τ_0, β)-KMS state on $\mathcal{O}_\vartheta$.

Let $V \in \mathcal{O}_\vartheta$ be a self-adjoint perturbation and τ_λ the perturbed C^*-dynamics generated by $\delta_\lambda = \delta_0 + \mathrm{i}\lambda[V, \,\cdot\,]$ where $\lambda \in \mathbb{R}$ is a coupling constant. We recall that for $A \in \mathcal{O}$ and $t \geq 0$,

$$\tau_\lambda^t(A) = \tau_0^t(A) + \sum_{n=1}^{\infty} (\mathrm{i}\lambda)^n \int\limits_{0 \leq s_n \leq \cdots \leq s_1 \leq t} [\tau_0^{s_n}(V), [\cdots, [\tau_0^{s_1}(V), \tau_0^t(A)] \cdots]] \, \mathrm{d}s_1 \cdots \mathrm{d}s_n. \tag{1.1}$$

In this paper we shall consider self-adjoint perturbations of the form

$$V = \sum_{k=1}^{K} \prod_{j=1}^{n_k} a^*(u_{kj}) a(v_{kj}), \tag{1.2}$$

where K and n_k's are finite. We set $\overline{n} = \max_k n_k$. Denote $\mathcal{D}_0 = \{u_{kj}, v_{kj}\}$. By rescaling λ, without loss of generality we may assume that

$$\max_{f \in \mathcal{D}_0} \|f\| = 1. \tag{1.3}$$

If $\overline{n} = 1$, then $\tau_\lambda^t(a^\#(f)) = a^\#(\mathrm{e}^{\mathrm{i}th_\lambda} f)$ where $h_\lambda = h_0 + \lambda \sum_k (v_k, \,\cdot\,) u_k$, and so the C^*-dynamics τ_λ is also a group of Bogoliubov $*$-automorphisms. This special case is exactly solvable and has been studied in detail in [AJPP2] (for additional information and references about quasi-free open quantum systems we refer the reader to recent lecture notes [AJPP1, JKP]).

The following technical result will play a key role in our paper.

Theorem 1.1 *Let $A = a^\#(f_1) \cdots a^\#(f_m)$ be a monomial of order m and*

$$\mathcal{C}_A^{(n)}(s_0, \ldots, s_n) = [V, [\tau_0^{s_n}(V), [\cdots, [\tau_0^{s_1}(V), \tau_0^{s_0}(A)] \cdots]]].$$

Then for all $n \geq 0$ there exist a finite index set $\mathcal{P}_n(A)$, monomials $F_{A,p}^{(n)} \in \mathcal{O}$, and scalar functions $G_{A,p}^{(n)}$ such that

$$\mathcal{C}_A^{(n)}(s_0, \ldots, s_n) = \sum_{p \in \mathcal{P}_n(A)} G_{A,p}^{(n)}(s_0, \ldots, s_n) F_{A,p}^{(n)}(s_0, \ldots, s_n). \tag{1.4}$$

Moreover,

1. *The order of the monomial $F_{A,p}^{(n)}$ does not exceed $2(n+1)(\overline{n}-1) + m$.*

2. *The factors of $F_{A,p}^{(n)}$ are from*

$$\{a^\#(\mathrm{e}^{\mathrm{i}sh_0} g) \mid g \in \mathcal{D}_0, s \in \{0, s_1, \ldots, s_n\}\} \cup \{a^\#(\mathrm{e}^{\mathrm{i}s_0 h_0} g) \mid g \in \mathcal{A}\},$$

where $\mathcal{A} = \{f_1, \cdots, f_m\}$. The number of factors from the first set does not exceed $(n+1)(2\overline{n}-1)$ while the number of factors from the second set does not exceed $m-1$.

3. *Suppose that*

$$\ell = \int_0^\infty \sup_{f \in \mathcal{D}_0, g \in \mathcal{D}_0 \cup \mathcal{A}} |(f, \mathrm{e}^{ith_0} g)| \, \mathrm{d}t < \infty,$$

denote

$$\ell_0 = \int_0^\infty \sup_{f, g \in \mathcal{D}_0} |(f, \mathrm{e}^{\mathrm{i}sh_0} g)| \, \mathrm{d}s,$$

and set

$$\Lambda_0 = \begin{cases} \dfrac{1}{2K\ell_0} & \text{if } \overline{n} = 1, \\ \dfrac{1}{2\overline{n}K\ell_0} \dfrac{(2\overline{n}-2)^{2\overline{n}-2}}{(2\overline{n}-1)^{2\overline{n}-1}} & \text{if } \overline{n} > 1. \end{cases}$$

If $\overline{n} = 1$ *and* $|\lambda| < \Lambda_0$ *or if* $\overline{n} > 1$ *and* $|\lambda| \le \Lambda_0$ *then the sum*

$$W = \sum_{n=0}^{\infty} |\lambda|^{n+1} \sum_{p\in\mathcal{P}_n(A)} \int_{0\le s_n\le\cdots\le s_0<\infty} |G_{A,p}^{(n)}(s_0,\ldots,s_n)|\,\mathrm{d}s_0\cdots\mathrm{d}s_n,$$

is finite and satisfies

$$W \le \left(1 + \frac{2\overline{n}K\ell|\lambda|}{(1-|\lambda|/\Lambda_0) + 2\overline{n}(2\overline{n}-2)K\ell_0|\lambda|}\right)^m - 1. \tag{1.5}$$

Remark 1. Parts (1) and (2) of this theorem are easy to prove and are stated for reference purpose. The key fact is Part (3) which we shall prove using the fundamental Botvich-Gută -Maassen integral estimate [BGM]. Related but weaker results can be obtained using the integral estimates of [BM1, BM2, FMU].
Remark 2. In our applications we shall not need the explicit form of the bound (1.5).

Our first regularity assumption is

(A1) There exists a dense vector subspace $\mathcal{D} \subset \mathfrak{h}$ such that $\mathcal{D}_0 \subset \mathcal{D}$ and that the functions

$$\mathbb{R} \ni t \mapsto (f, \mathrm{e}^{\mathrm{i}th_0} g),$$

are in $L^1(\mathbb{R}, \mathrm{d}t)$ for all $f, g \in \mathcal{D}$.

Note that this assumption implies that h_0 has purely absolutely continuous spectrum.
A consequence of Theorem 1.1 is

Theorem 1.2 *Assume that (A1) holds and that* $|\lambda| < \Lambda_0$. *Then the limits*

$$\gamma_\lambda^+(A) = \lim_{t\to+\infty} \tau_0^{-t} \circ \tau_\lambda^t(A), \tag{1.6}$$

exist for all $A \in \mathcal{O}$ *and define a* $*$*-automorphism* $\gamma_\lambda^+ : \mathcal{O} \to \mathcal{O}$.

Remark. Under additional regularity assumptions one can also obtain information about the rate of convergence in (1.6), see [JP4] for details.

Although Theorem 1.2 is a well-known result (see [Ro, BM1, BM2, FMU]), for the reader convenience we will sketch its proof in Subsection 2.2.

1.2 The model and the result

Our starting point are finitely many, say M, independent free Fermi gasses $\mathcal{R}_j$ in equilibrium at inverse temperatures $\beta_j > 0$ and chemical potentials $\mu_j \in \mathbb{R}$. More precisely, $\mathcal{R}_j$ is described by the quantum dynamical system $(\mathcal{O}_j, \tau_j, \omega_j)$ where:

(i) $\mathcal{O}_j = \mathrm{CAR}(\mathfrak{h}_j)$ is the CAR algebra over the single fermion Hilbert space $\mathfrak{h}_j$;

(ii) τ_j^t is the group of Bogoliubov $*$-automorphisms generated by the single fermion Hamiltonian h_j;

(iii) ω_j is the gauge-invariant quasi-free state generated by

$$T_j = \frac{1}{1 + \mathrm{e}^{\beta_j(h_j - \mu_j)}}.$$

We denote by ϑ_j the gauge group of $\mathcal{R}_j$. The generators of τ_j and ϑ_j are denoted by δ_j and ξ_j.

Let

$$\mathfrak{h} = \bigoplus_{j=1}^{M} \mathfrak{h}_j, \qquad h_0 = \bigoplus_{j=1}^{M} h_j, \qquad T = \bigoplus_{j=1}^{M} T_j.$$

The joint system $\mathcal{R} = \sum \mathcal{R}_j$ in absence of interaction is described by the quantum dynamical system $(\mathcal{O}, \tau_0, \omega)$, where $\mathcal{O} = \mathrm{CAR}(\mathfrak{h})$, τ_0^t is the group of Bogoliubov $*$-automorphisms generated by h_0, and ω is the gauge-invariant quasi-free state generated by T. We denote by δ_0 the generator of τ_0 and by ξ the generator of the gauge group ϑ of the joint system. Obviously, $\delta_0 = \sum_j \delta_j$ and $\xi = \sum_j \xi_j$.

Let $V \in \mathcal{O}_\vartheta$ be a perturbation of the form (1.2). This perturbation describes the coupling of the reservoirs, and, possibly, self-interactions within the reservoirs. Let $\lambda \in \mathbb{R}$ be a coupling constant and τ_λ the C^*-dynamics on $\mathcal{O}$ generated by $\delta_\lambda = \delta_0 + \mathrm{i}\lambda[V, \cdot\,]$. The interacting joint system is described by the quantum dynamical system $(\mathcal{O}, \tau_\lambda, \omega)$.

Let γ_λ^+ be as in Theorem 1.2 and $\omega_{\lambda+} = \omega \circ \gamma_\lambda^+$. A consequence of Theorem 1.2 (see Subsection 2.2) is:

Theorem 1.3 *Assume that (A1) holds and that $|\lambda| < \Lambda_0$. Then for all ω-normal states η and $A \in \mathcal{O}$,*

$$\lim_{t \to +\infty} \eta \circ \tau_\lambda^t(A) = \omega_{\lambda+}(A).$$

The state $\omega_{\lambda+}$ is the NESS of the quantum dynamical system $(\mathcal{O}, \tau_\lambda, \omega)$ [Ru1, JP3]. Clearly, this NESS depends on β_j and μ_j.

Let $\beta_{\mathrm{eq}} > 0$ and $\mu_{\mathrm{eq}} \in \mathbb{R}$ be given (equilibrium) values of the inverse temperature and chemical potential. We are interested in linear response of $\mathcal{R}$ to thermodynamical forces

$$X_j = \beta_{\mathrm{eq}} - \beta_j, \qquad Y_j = \beta_j \mu_j - \beta_{\mathrm{eq}} \mu_{\mathrm{eq}}.$$

Let $X = (X_1, \cdots, X_M)$, $Y = (Y_1, \cdots, Y_M)$. We indicate the dependence on X, Y by denoting

$$\omega_{XY} = \omega, \qquad \omega_{\lambda XY+} = \omega_{\lambda+}, \qquad T_{XY} = T.$$

Note that by Araki perturbation theory $\omega_{\lambda 00+}$ is the unique β_{eq}-KMS state for the C^*-dynamics $\tau_\lambda^t \circ \vartheta^{-\mu_{\mathrm{eq}} t}$. We denote this state by $\omega_{\lambda \mathrm{eq}}$.

In what follows we shall assume:

(A2) The operators h_j are bounded.

Although our method of proof extends to unbounded h_j's (see Remark 2 after Theorem 1.5), the above assumption covers most cases of physical interest to which our results apply and allows for technically simpler exposition of the proofs.

The observables describing the heat and charge flux out of $\mathcal{R}_j$ are

$$\Phi_j = \lambda \delta_j(V), \qquad \mathcal{J}_j = \lambda \xi_j(V). \tag{1.7}$$

Clearly, $\Phi_j, \mathcal{J}_j \in \mathcal{O}_\vartheta$. The conservation laws

$$\sum_{j=1}^{M} \omega_{\lambda XY+}(\Phi_j) = 0, \qquad \sum_{j=1}^{M} \omega_{\lambda XY+}(\mathcal{J}_j) = 0,$$

hold. The entropy production of the NESS $\omega_{\lambda XY+}$ is defined by

$$\mathrm{Ep}(\omega_{\lambda XY+}) = \omega_{\lambda XY+}\left(-\sum_{j=1}^{M}\beta_j\left(\Phi_j - \mu_j \mathcal{J}_j\right)\right) = \sum_{j=1}^{M} X_j \omega_{\lambda XY+}(\Phi_j) + \sum_{j=1}^{M} Y_j \omega_{\lambda XY+}(\mathcal{J}_j).$$

By the general results of [Ru2, JP2] (see also [TM, FMU, JOP2]), $\mathrm{Ep}(\omega_{\lambda XY+}) \geq 0$. The strict positivity of the entropy production for locally interacting fermionic reservoirs can be established by using either perturbative arguments (see [FMU]) or stability arguments (see Section 4.3 in [JP3] and [JP4]). This point is discussed in more detail in the forthcoming review [JP5].

To study linear response of $\omega_{\lambda XY+}$, in addition to (A1)-(A2) we need the following regularity assumption.

(A3) For all j and $g \in \mathcal{D}_0$, $h_j g \in \mathcal{D}$.

Our final assumption concerns time-reversal invariance.

(A4) There exists a complex conjugation c on $\mathfrak{h}$ which commutes with all h_j and satisfies $cg = g$ for all $g \in \mathcal{D}_0$.

If (A4) holds, then the map $\Theta(a^{\#}(f)) = a^{\#}(cf)$ extends to an involutive skew $*$-automorphism of $\mathcal{O}$ such that $\Theta \circ \tau_j^t = \tau_j^{-t} \circ \Theta$ and $\Theta(V) = V$. This implies that $\Theta \circ \tau_\lambda^t = \tau_\lambda^{-t} \circ \Theta$ for all λ. Note also that

$$\Theta(\Phi_j) = -\Phi_j, \qquad \Theta(\mathcal{J}_j) = -\mathcal{J}_j.$$

We set

$$\mathbb{I}_\epsilon = \{(X,Y) \in \mathbb{R}^{2M} \,|\, |X_j| < \epsilon,\, |Y_j| < \epsilon\},$$

$$D_\epsilon = \{(X,Y) \in \mathbb{C}^{2M} \,|\, |X_j| < \epsilon,\, |Y_j| < \epsilon\},$$

$$R_{\Lambda,\delta} = \{\lambda \in \mathbb{C} \,|\, |\mathrm{Re}\,\lambda| < \Lambda, |\mathrm{Im}\,\lambda| < \delta\}.$$

In the sequel $\mathfrak{F}_j$ stands for either Φ_j or $\mathcal{J}_j$. Our first result is:

Theorem 1.4 *Suppose that Assumptions (A1)-(A3) hold and let* $0 < \Lambda < \Lambda_0$. *Then there exist* $\epsilon > 0$ *and* $\delta > 0$ *such that the maps*

$$(\lambda, X, Y) \mapsto \omega_{\lambda XY+}(\mathfrak{F}_j),$$

extend to analytic functions on the set $R_{\Lambda,\delta} \times D_\epsilon$. *In particular, for any* $|\lambda| < \Lambda_0$ *there exists* $\epsilon(\lambda) > 0$ *such that the maps*

$$(X,Y) \mapsto \omega_{\lambda XY+}(\mathfrak{F}_j),$$

extend to analytic functions on $D_{\epsilon(\lambda)}$.

The kinetic transport coefficients are defined by

$$\begin{aligned}
L^{kj}_{\lambda\mathrm{hh}} &= \partial_{X_j}\omega_{\lambda XY+}(\Phi_k)\big|_{X=Y=0},\\
L^{kj}_{\lambda\mathrm{hc}} &= \partial_{Y_j}\omega_{\lambda XY+}(\Phi_k)\big|_{X=Y=0},\\
L^{kj}_{\lambda\mathrm{ch}} &= \partial_{X_j}\omega_{\lambda XY+}(\mathcal{J}_k)\big|_{X=Y=0},\\
L^{kj}_{\lambda\mathrm{cc}} &= \partial_{Y_j}\omega_{\lambda XY+}(\mathcal{J}_k)\big|_{X=Y=0},
\end{aligned} \tag{1.8}$$

where the indices h/c stand for heat/charge. For $A, B \in \mathcal{O}_\vartheta$ we set

$$\mathcal{L}_\lambda(A,B) = \lim_{t\to+\infty} \frac{1}{2}\int_{-t}^{t} \omega_{\lambda\mathrm{eq}}(\tau_\lambda^s(A)B)\,\mathrm{d}s,$$

and

$$\mathfrak{L}_\lambda(A,B) = \lim_{t\to+\infty} \frac{1}{\beta_{\mathrm{eq}}}\int_0^t \mathrm{d}s \int_0^{\beta_{\mathrm{eq}}} \mathrm{d}u\, \omega_{\lambda\mathrm{eq}}(\tau_\lambda^s(A)\tau_\lambda^{\mathrm{i}u}(B)),$$

whenever the limits exist. Our main result is:

Theorem 1.5 *Suppose that Assumptions (A1)-(A3) hold and that* $|\lambda| < \Lambda_0$. *Then* $\mathfrak{L}_\lambda(A,B)$ *is well-defined for* $A, B \in \{\Phi_1, \cdots, \Phi_M, \mathcal{J}_1, \cdots, \mathcal{J}_M\}$ *and*

$$\begin{aligned} L_{\lambda\mathrm{hh}}^{kj} &= \mathfrak{L}_\lambda(\Phi_k, \Phi_j), \\ L_{\lambda\mathrm{hc}}^{kj} &= \mathfrak{L}_\lambda(\Phi_k, \mathcal{J}_j), \\ L_{\lambda\mathrm{ch}}^{kj} &= \mathfrak{L}_\lambda(\mathcal{J}_k, \Phi_j), \\ L_{\lambda\mathrm{cc}}^{kj} &= \mathfrak{L}_\lambda(\mathcal{J}_k, \mathcal{J}_j). \end{aligned} \tag{1.9}$$

Assume in addition that (A4) holds. Then $\mathcal{L}_\lambda(A,B)$ *is well-defined for* $A, B \in \{\Phi_1 \cdots \Phi_M, \mathcal{J}_1, \cdots \mathcal{J}_M\}$,

$$\begin{aligned} L_{\lambda\mathrm{hh}}^{kj} &= \mathcal{L}_\lambda(\Phi_k, \Phi_j), \\ L_{\lambda\mathrm{hc}}^{kj} &= \mathcal{L}_\lambda(\Phi_k, \mathcal{J}_j), \\ L_{\lambda\mathrm{ch}}^{kj} &= \mathcal{L}_\lambda(\mathcal{J}_k, \Phi_j), \\ L_{\lambda\mathrm{cc}}^{kj} &= \mathcal{L}_\lambda(\mathcal{J}_k, \mathcal{J}_j), \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} L_{\lambda\mathrm{hh}}^{kj} &= L_{\lambda\mathrm{hh}}^{jk}, \\ L_{\lambda\mathrm{cc}}^{kj} &= L_{\lambda\mathrm{cc}}^{jk}, \\ L_{\lambda\mathrm{hc}}^{kj} &= L_{\lambda\mathrm{ch}}^{jk}. \end{aligned} \tag{1.11}$$

Remark 1. The formulas (1.9) are the GKF without time reversal assumption. The formulas (1.10) are the GKF in the standard form. The formulas (1.11) are the Onsager reciprocity relations. The ORR are an immediate consequence of (1.10) and the KMS condition, see [JOP1, JOP2].

Remark 2. If $\overline{n} = 1$, then our proofs give that Theorems 1.1-1.5 hold with $\Lambda_0 = 1/2K\ell_0$. However, since in this case the coupled system is quasi-free, these theorems can be also proven using trace class scattering theory which yields better constants and wealth of additional information about the model. For more information about this special case we refer the reader to [AJPP1, AJPP2, JKP].

Remark 3. With regard to the Green-Kubo formulas (1.10), a natural question is whether the correlation functions $t \mapsto \omega_{\lambda\mathrm{eq}}(\tau_\lambda^t(A)B)$ are absolutely integrable for $A, B \in \{\Phi_1, \cdots, \Phi_M, \mathcal{J}_1, \cdots \mathcal{J}_M\}$. This is a delicate dynamical problem which is studied in [JPP]. In this paper we only establish the existence of the improper integrals

$$\lim_{t\to+\infty} \int_{-t}^{t} \omega_{\lambda\mathrm{eq}}(\tau_\lambda^s(A)B)\,\mathrm{d}s.$$

Remark 4. By Theorem 1.4, the functions $\lambda \mapsto L^{kj}_{\lambda \mathrm{uv}}$, $\mathrm{u}, \mathrm{v} \in \{\mathrm{h}, \mathrm{c}\}$, are analytic for $|\lambda| < \Lambda_0$ and can be expanded into power series whose coefficients can be computed. Such computations can be used to verify that in specific examples the transport coefficients are non-vanishing. For reasons of space we shall discuss these perturbative computations in the forthcoming review [JP5].
Remark 5. Our results are tailored for application to tight-binding type models of electronic transport in which Assumption (A2) is usually satisfied. However, all our proofs extend to unbounded h_j's as long as $\mathcal{D}_0 \subset \mathrm{Dom}\,(\mathrm{e}^{a|h_j|})$ for all j and some $a > \beta_{\mathrm{eq}}$. It is an interesting technical problem to prove Theorems 1.4 and 1.5 for unbounded h_j's without this additional technical assumption.
Remark 6. Theorems 1.1-1.3 are fairly flexible and are easily adapted to a number of different setups involving free Fermi gas reservoirs. The same applies to Theorems 1.4 and 1.5. For example, one may consider the tensor product structure, where the joint system in absence of interaction is described by $\mathcal{O} = \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}_M$, $\tau_0 = \tau_1 \otimes \cdots \otimes \tau_M$, $\omega = \omega_1 \otimes \cdots \otimes \omega_M$. This type of models was studied in [FMU]. Another class of related models are local perturbations of the exactly solvable Electronic Black-Box Model studied in [AJPP1, AJPP2]. Instead of coupled free fermionic systems one may consider coupled $\mathrm{X} - \mathrm{Y}$ quantum spin chains. Theorems 1.4 and 1.5 extend to these models with only notational changes, see [JP5] for details.
Remark 7. We call $A \in \mathcal{O}$ centered if $\omega_{\lambda XY}(A) = 0$ for all $|\lambda| < \Lambda_0$ and $(X, Y) \in \mathbb{I}_\epsilon$. Our proof easily extends to the general Green–Kubo formulas

$$\partial_{X_j} \omega_{\lambda XY+}(A)\big|_{X=Y=0} = \mathcal{L}_\lambda(A, \Phi_j), \quad \partial_{Y_j} \omega_{\lambda XY+}(A)\big|_{X=Y=0} = \mathcal{L}_\lambda(A, \mathcal{J}_j),$$

for centered observables A which are polynomials in $a^\#(f)$ with $f \in \mathcal{D}$.

We finish this subsection with some examples to which Theorems 1.1-1.5 apply. Let $\mathcal{G}$ be the set of vertices of a connected graph of bounded degree and $\Delta_{\mathcal{G}}$ the associated discrete Laplacian acting on $l^2(\mathcal{G})$. We recall that

$$(\Delta_{\mathcal{G}} \psi)(x) = \sum_{|y-x|=1} \psi(y),$$

where $|y-x|$ is the distance on the graph. $\Delta_{\mathcal{G}}$ is a bounded self-adjoint operator and $\|\Delta_{\mathcal{G}}\| = \sup_{x \in \mathcal{G}} d(x)$, where $d(x)$ is the degree of the vertex x. Let δ_x be the Kronecker delta function at $x \in \mathcal{G}$. We shall call the graph $\mathcal{G}$ *admissible* if there exists $\gamma > 1$ such that for all $x, y \in \mathcal{G}$,

$$|(\delta_x, \mathrm{e}^{-\mathrm{i}t\Delta_{\mathcal{G}}} \delta_y)| = O(|t|^{-\gamma}), \tag{1.12}$$

as $t \to \infty$. Clearly, the discrete Laplacian of an admissible graph has purely absolutely continuous spectrum.

An example of admissible graph is $\mathcal{G} = \mathbb{Z}^d$ for $d \geq 3$. In this case $\gamma = d/2$. Another example is the half-space $\mathcal{G} = \mathbb{Z}_+ \times \mathbb{Z}^{d-1}$ where $\mathbb{Z}_+ = \{0, 1, \cdots\}$ and $d \geq 1$ (if $d = 1$ then $\mathcal{G} = \mathbb{Z}_+$). In this case $\gamma = (d+2)/2$. Tubular graphs of the type $\mathbb{Z}_+ \times \Gamma$, where $\Gamma \subset \mathbb{Z}^{d-1}$ is finite, are admissible with $\gamma = 3/2$. Another well-known admissible graph is a rooted Bethe lattice where $\gamma = 3/2$.

Assumptions (A1)–(A4) and Theorems 1.1-1.5 hold if

(i) $\mathcal{G}_1, \ldots, \mathcal{G}_M$ are admissible graphs;

(ii) $\mathfrak{h}_j = \ell^2(\mathcal{G}_j)$ or more generally $\ell^2(\mathcal{G}_j) \otimes \mathbb{C}^L$ to allow for internal degrees of freedom (*e.g.,* spin);

(iii) $\mathcal{D}$ is the subspace of finitely supported elements of $\mathfrak{h}$;

(iv) $h_j = -\Delta_{\mathcal{G}_j}$;

(v) u_{kj}, v_{kj} belong to $\mathcal{D}$.

Allowed interactions include $V = V^{\mathrm{hop}} + V^{\mathrm{int}}$ where

(i) $V^{\rm hop}$ describes tunneling junctions between the reservoirs:

$$V^{\rm hop} = \sum_{x,y} t(x,y)\left(a^*(\delta_x)a(\delta_y) + a^*(\delta_y)a(\delta_x)\right),$$

where $t : \mathcal{G}\times\mathcal{G}\to\mathbb{R}$ is a finitely supported function ($\mathcal{G} = \cup_j\mathcal{G}_j$);

(ii) $V^{\rm int}$ is a local pair interaction

$$V^{\rm int} = \sum_{x,y} v(x,y)a^*(\delta_x)a^*(\delta_y)a(\delta_y)a(\delta_x),$$

where $v : \mathcal{G}\times\mathcal{G}\to\mathbb{R}$ is finitely supported.

This concrete model is studied in detail in [JP5].

2 Basic properties of the model

In this section we prove Theorems 1.1, 1.2, and 1.3.

2.1 Proof of Theorem 1.1

We start with some preliminaries which are of independent interest. Let $A = a_1\cdots a_m$ and $B = b_1\cdots b_q$ where the a_k and b_j are creation/annihilation operators. Thus, A and B are monomials of order m and q respectively. If q is even it follows from the CAR that

$$\begin{aligned}
[B,a_j] &= b_1\cdots b_q a_j - a_j b_1\cdots b_q\\
&= b_1\cdots b_q a_j - (\{b_1,a_j\} - b_1 a_j)b_2\cdots b_q\\
&= -\{b_1,a_j\}b_2\cdots b_q + b_1(b_2\cdots b_q a_j + a_j b_2\cdots b_q)\\
&\ \ \vdots\\
&= \sum_{k=1}^{q}(-1)^k\{b_k,a_j\}b_1\cdots b_{k-1}b_{k+1}\cdots b_q,
\end{aligned}$$

and hence

$$\begin{aligned}
[B,A] &= \sum_{j=1}^{m} a_1\cdots a_{j-1}[B,a_j]a_{j+1}\cdots a_m\\
&= \sum_{j=1}^{m}\sum_{k=1}^{q}(-1)^k\{b_k,a_j\}a_1\cdots a_{j-1}b_1\cdots b_{k-1}b_{k+1}\cdots b_n a_{j+1}\cdots a_m.
\end{aligned}$$

The anticommutator $\{b_k,a_j\}$ on the right hand side is called contraction of the factor b_k of B with the factor a_j of A. Note that contractions are numbers.

Iterating the last formula we get, for any monomials $B_1, B_2, \ldots B_{n+1}$ of even orders $q_1, q_2, \ldots q_{n+1}$ and any monomial A of order m

$$[B_{n+1},[\tau_0^{s_n}(B_n),[\cdots,[\tau_0^{s_1}(B_1),\tau_0^{s_0}(A)]\cdots]]] = \sum_{p\in\mathcal{P}_n(A,B_1,\ldots,B_{n+1})} G_p(s_0,\cdots,s_n)F_p, \tag{2.13}$$

where the F_p are monomials of order $q_1+q_2+\cdots+q_{n+1}+m-2(n+1)$ and the coefficients G_p are products of $n+1$ contractions. The sum on the right hand side runs over the set $\mathcal{P}_n(A,B_1,\ldots,B_{n+1})$ whose elements p are contraction diagrams of the type displayed in Figure 1.

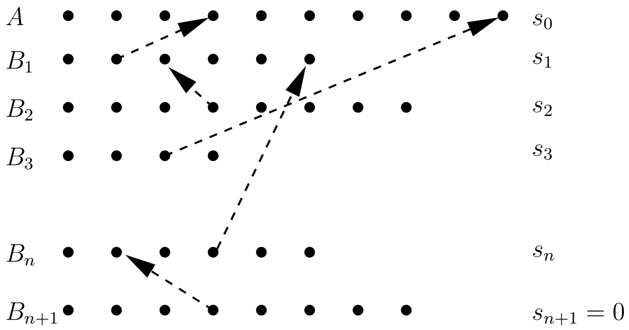

Figure 1: An element of the set $\mathcal{P}_n$.

Each line of this diagram represents a monomial, as labeled on the left. Each dot on a line represents a factor of the corresponding monomial. The dashed lines represent contractions of such factors. From each line of the diagram there is exactly one contraction going up and any factor can belong only to one contraction. To a contraction diagram p we associate its skeleton: a rooted tree T whose nodes are 0 (the root), $1, \ldots, n+1$ and whose bonds correspond to the contractions (see Figure 2). The skeleton T is simply obtained by collapsing each line of the contraction diagram p to a single node. If there is an arrow going from the node j to the node k in T we say that j is a child of k or that k is the parent of j (each node has a unique parent and we shall say that the root node 0 is its own parent). We can describe the rooted tree T by the function $T\colon \{0, \ldots, n+1\} \to \{0, \ldots, n+1\}$ which to a node j associates its parent $T(j)$. Reciprocally, any function T such that $T(0) = 0$ and $T(j) < j$ for $j = 1, \ldots, n+1$ defines a rooted tree T. Such a function is called a climber of order $n+1$ and there is a one-to-one correspondence between climbers and rooted trees.

Suppose that all the factors of the monomials $B_1, \ldots, B_{n+1}$ are from $\{a^\#(g) \mid g \in \mathcal{D}_0\}$ and let A and $\mathcal{A}$ be as in Theorem 1.1. Then, the factors of the monomials F_p are from

$$\{a^\#(\mathrm{e}^{\mathrm{i}sh_0}g) \mid g \in \mathcal{D}_0, s \in \{0, s_1, \ldots, s_n\}\} \cup \{a^\#(\mathrm{e}^{\mathrm{i}s_0h_0}g) \mid g \in \mathcal{A}\}.$$

The number of factors from the first set does not exceed $(n+1)(\overline{q}-1)$, where $\overline{q} = \max q_k$. The number of factors from the second set does not exceed $m-1$. If we denote

$$S_k(t) \equiv \begin{cases} \sup\limits_{f\in\mathcal{D}_0, g\in\mathcal{A}} |(f, \mathrm{e}^{\mathrm{i}th_0}g)| & \text{for } k = 0, \\ \sup\limits_{f,g\in\mathcal{D}_0} |(f, \mathrm{e}^{\mathrm{i}th_0}g)| & \text{for } k > 0, \end{cases}$$

then all coefficients G_p associated with a given skeleton tree T are bounded by

$$|G_p| \leqslant \prod_{j=1}^{n+1} S_{T(j)}(s_{T(j)} - s_j) \equiv S(T),$$

where we set $s_{n+1} = 0$. Thus, if $N(T)$ denotes the number of contraction diagrams with skeleton tree T we have

$$\sum_{p\in\mathcal{P}_n} |G_p| \leqslant \sum_{T\in\mathcal{T}_{n+1}} N(T)S(T),$$

where $\mathcal{T}_{n+1}$ denotes the set of all rooted trees with nodes 0 (the root), $1, \ldots, n+1$. Let us compute $N(T)$. To this end denote by r_j the number of childs of the node j. For the tree of Figure 2 we have for example $r_0 = 2$, $r_1 = 2$, $r_2 = r_3 = 0$, $r_n = 1$ and $r_{n+1} = 0$. Clearly, $N(T) = 0$ if $r_0 > m$ or $r_j > q_j - 1$. Otherwise, to construct a

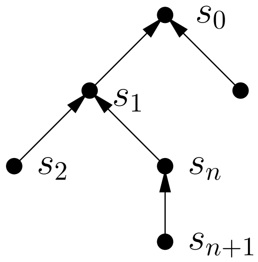

Figure 2: The skeleton tree corresponding to Figure 1.

diagram p whose skeleton is T we first have to choose a factor on each line $B_1, \ldots, B_{n+1}$. The number of such choices is clearly $q_1 q_2 \cdots q_{n+1}$. Now on line A we have to choose one factor for each of the r_0 childs of node 0. There are $m(m-1)\cdots(m-r_0+1)$ such choices. Similarly, on line B_1 we have to chose r_1 factors out of the $q_1 - 1$ remaining. There are $(q_1-1)(q_1-2)\cdots(q_1-r_1)$ such choices. The same reasoning applies to lines $B_2, \ldots, B_n$, and we conclude that

$$N(T) = \frac{m!}{(m-r_0)!}\prod_{j=1}^{n}\frac{q_j!}{(q_j-r_j-1)!} \le \frac{m!}{(m-r_0)!}\prod_{j=1}^{n}\frac{\overline{q}!}{(\overline{q}-r_j-1)!} \equiv \overline{N}(T).$$

We now turn to the proof of Theorem 1.1. Since $V = \sum_{k=1}^{K} V_k$ where V_k are monomials of order $q_k = 2n_k$ (so $\overline{q} = 2\overline{n}$), we can write

$$\mathcal{C}_A^{(n)}(s_0, \ldots, s_n) = \sum_{k_1,\ldots,k_{n+1}=1}^{K} [V_{k_{n+1}}, [\tau_0^{s_n}(V_{k_n}), [\cdots, [\tau_0^{s_1}(V_{k_1}), \tau_0^{s_0}(A)]\cdots]]],$$

and Parts (1) and (2) follow immediately with

$$\mathcal{P}_n(A) \equiv \bigcup_{k_1,\ldots,k_{n+1}=1}^{K} \{(k_1, \ldots, k_{n+1})\} \times \mathcal{P}_n(A, V_{k_1}, \ldots, V_{k_{n+1}}).$$

To prove (3), we start with the estimate

$$\sum_{p\in\mathcal{P}_n(A)} |G_{A,p}^{(n)}| \le K^{n+1} \sum_{T\in\mathcal{T}_{n+1}} \overline{N}(T)S(T).$$

Hence,

$$W \equiv \sum_{n=0}^{\infty} |\lambda|^{n+1} \sum_{p\in\mathcal{P}_n(A)} \int_{0\le s_n\le\cdots\le s_0} |G_{A,p}^{(n)}(s_0, \ldots, s_n)|\, \mathrm{d}s_0\cdots\mathrm{d}s_n,$$

satisfies

$$W \le W_0 \equiv \sum_{n=1}^{\infty}\sum_{T\in\mathcal{T}_n} \overline{N}(T) \int_{0=s_n\le s_{n-1}\cdots\le s_0} \prod_{j=1}^{n}(|\lambda| K S_{T(j)}(s_{T(j)} - s_j))\, \mathrm{d}s_0\cdots\mathrm{d}s_{n-1}.$$

We will need the following general result of [BGM].

Theorem 2.1 *Let $m_k, \widetilde{m}_k$ be two sequences of nonnegative numbers and $g, \widetilde{g}$ two integrable nonnegative functions on $[0,\infty[$. Denote by $\|g\|$ and $\|\widetilde{g}\|$ their L^1-norms, set $g_0 = \widetilde{g}$ and $g_k = g$ for $k > 0$ and define*

$$M(x) \equiv \sum_{k=0}^{\infty} \frac{m_k}{k!} x^k, \quad \widetilde{M}(x) \equiv \sum_{k=0}^{\infty} \frac{\widetilde{m}_k}{k!} x^k.$$

To any rooted tree $T \in \mathcal{T}_n$ associate the weight (recall that r_j is the number of childs of the node j),

$$w(T) = \widetilde{m}_{r_0} m_{r_1} \cdots m_{r_n} \int_{0=s_n \le s_{n-1} \le \cdots \le s_0} \prod_{j=1}^{n} g_{T(j)}(s_{T(j)} - s_j)\, \mathrm{d}s_0 \cdots \mathrm{d}s_{n-1}.$$

Then, the sum $W = \sum_{n=1}^{\infty} \sum_{T \in \mathcal{T}_n} w(T)$ is finite if and only if the equation $M(\|g\|x) = x$ has a positive solution x such that $\widetilde{M}(\|\widetilde{g}\|x) < \infty$. If x^* denotes the least such solution, then $W = \widetilde{M}(\|\widetilde{g}\|x^*)$.

To apply this result we set $\widetilde{m}_k = 0$ for $k = 0$ and $k > m$, otherwise

$$\widetilde{m}_k = \frac{m!}{(m-k)!},$$

$m_k = 0$ for $k \ge 2\overline{n}$, otherwise

$$m_k = \frac{(2\overline{n})!}{(2\overline{n} - k - 1)!},$$

and

$$g(s) = |\lambda| K S_1(s), \quad \widetilde{g}(s) = |\lambda| K S_0(s).$$

Hence, $M(x) = 2\overline{n}(1+x)^{2\overline{n}-1}$, $\widetilde{M}(x) = (1+x)^m - 1$, $\|g\| = |\lambda| K \ell_0$, and $\|\widetilde{g}\| = |\lambda| K \ell$. An elementary analysis shows that, if

$$\Lambda_0 = \begin{cases} \dfrac{1}{2K\ell_0} & \text{for } \overline{n} = 1, \\ \dfrac{1}{2\overline{n} K \ell_0} \dfrac{(2\overline{n}-2)^{2\overline{n}-2}}{(2\overline{n}-1)^{2\overline{n}-1}} & \text{for } \overline{n} > 1, \end{cases},$$

then, as long as $|\lambda| < \Lambda_0$ for $\overline{n} = 1$ and $|\lambda| \le \Lambda_0$ for $\overline{n} > 1$, the equation $M(\|g\|x) = x$ has a least positive solution x^* satisfying

$$0 \le x^* \le \frac{2\overline{n}}{(1 - |\lambda|/\Lambda_0) + 2\overline{n}(2\overline{n}-2) K \ell_0 |\lambda|},$$

and that

$$W \le W_0 = (1 + K\ell|\lambda| x^*)^m - 1 \le \left(1 + \frac{2\overline{n} K \ell |\lambda|}{(1 - |\lambda|/\Lambda_0) + 2\overline{n}(2\overline{n}-2) K \ell_0 |\lambda|}\right)^m - 1.$$

This ends the proof of Theorem 1.1.

2.2 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. To establish the existence of the limit (1.6) for all $A \in \mathcal{O}$ it suffices to consider the case $A = a^{\#}(f)$ with $f \in \mathcal{D}$ and $\|f\| = 1$. Since

$$\tau_0^{-t_2} \circ \tau_\lambda^{t_2}(A) - \tau_0^{-t_1} \circ \tau_\lambda^{t_1}(A) = \mathrm{i}\lambda \int_{t_1}^{t_2} \tau_0^{-s}([V, \tau_\lambda^s(A)])\, \mathrm{d}s,$$

we have that

$$\|\tau_0^{-t_2} \circ \tau_\lambda^{t_2}(A) - \tau_0^{-t_1} \circ \tau_\lambda^{t_1}(A)\| \le |\lambda| \int_{t_1}^{t_2} \|[V, \tau_\lambda^s(A)]\| \, \mathrm{d}s. \tag{2.14}$$

The expansion (1.1) yields

$$[V, \tau_\lambda^s(A)] = [V, \tau_0^s(A)] + \sum_{n=1}^{\infty} (\mathrm{i}\lambda)^n \int\limits_{0 \le s_n \le \cdots \le s_1 \le s} [V, [\tau_0^{s_n}(V), [\cdots, [\tau_0^{s_1}(V), \tau_0^s(A)] \cdots]]] \, \mathrm{d}s_1 \cdots \mathrm{d}s_n.$$

Our standing assumption (1.3) and the fact that $\|f\| = 1$ implies that $\|F_{A,p}^{(n)}\| \le 1$ and we can estimate

$$\|[V, [\tau_0^{s_n}(V), [\cdots, [\tau_0^{s_1}(V), \tau_0^s(A)] \cdots]]]\| \le \sum_{p \in \mathcal{P}_n(A)} |G_{A,p}^{(n)}(s, s_1, \cdots, s_n)|.$$

Part (3) of Theorem 1.1 yields that for $|\lambda| < \Lambda_0$,

$$\int_0^\infty \|[V, \tau_\lambda^s(A)]\| \, \mathrm{d}s < \infty. \tag{2.15}$$

The estimates (2.14) and (2.15) imply the existence of the limit (1.6) for $|\lambda| < \Lambda_0$.

The map γ_λ^+ is obviously a $*$-morphism. To prove that it is an isomorphism, it suffices to show that the limits

$$\lim_{t \to +\infty} \tau_\lambda^{-t} \circ \tau_0^t(A),$$

exist for all $A \in \mathcal{O}$. Repeating the above argument we see that it suffices to show that

$$\int_0^\infty \|[V, \tau_0^s(A)]\| \, \mathrm{d}s < \infty,$$

for $A = a^\#(f)$, $f \in \mathcal{D}$. But this is a special case of Equ. (2.15). □

Proof of Theorem 1.3. Since h has purely absolutely continuous spectrum the quantum dynamical system $(\mathcal{O}, \tau_0, \omega)$ has the property of return to equilibrium: for all ω-normal states η and $A \in \mathcal{O}$,

$$\lim_{|t| \to \infty} \eta \circ \tau_0^t(A) = \omega(A),$$

see, e.g., [AJPP1]. The existence of norm-limits (1.6) ensures that

$$\lim_{t \to +\infty} \eta(\tau_\lambda^t(A)) = \lim_{t \to +\infty} \eta \circ \tau_0^t(\tau_0^{-t} \circ \tau_\lambda^t(A)) = \lim_{t \to +\infty} \eta \circ \tau_0^t(\gamma_\lambda^+(A)) = \omega(\gamma_\lambda^+(A)),$$

and the statement follows. □

3 Proofs of Theorems 1.4 and 1.5

3.1 Strategy

The strategy of the proofs of Theorems 1.4 and 1.5 is based on the arguments in [JOP3]. Consider the C^*-dynamics σ_{XY} on $\mathcal{O}$ generated by

$$\delta_{XY} = \delta_0 - \mu_{\mathrm{eq}} \xi - \sum_j \frac{X_j}{\beta_{\mathrm{eq}}} \delta_j - \sum_j \frac{Y_j}{\beta_{\mathrm{eq}}} \xi_j. \tag{3.16}$$

The reference state ω_{XY} is the unique $(\sigma_{XY}, \beta_{\rm eq})$-KMS state on $\mathcal{O}$. Let $\sigma_{\lambda XY}$ be the C^*-dynamics on $\mathcal{O}$ generated by

$$\delta_{\lambda XY} = \delta_{XY} + {\rm i}\lambda[V, \cdot].$$

The Araki perturbation theory [Ar, BR2, DJP] yields that there exists a unique $(\sigma_{\lambda XY}, \beta_{\rm eq})$-KMS state on $\mathcal{O}$. We denote this state by $\omega_{\lambda XY}$. The states ω_{XY} and $\omega_{\lambda XY}$ are mutually normal.

Recall that $\mathfrak{F}_j$ stands for either Φ_j or $\mathcal{J}_j$. Our main technical result is:

Theorem 3.1 *Suppose that Assumptions (A1)-(A3) hold and let* $0 < \Lambda < \Lambda_0$. *Then there exist* $\epsilon > 0$ *and* $\delta > 0$ *such that for all* $t \geq 0$ *the functions* $(\lambda, X, Y) \mapsto \omega_{\lambda XY}(\tau_\lambda^t(\mathfrak{F}_j))$ *have analytic extensions to* $R_{\Lambda,\delta} \times D_\epsilon$ *satisfying*

$$\sup_{\lambda \in R_{\Lambda,\delta}, (X,Y) \in D_\epsilon, t \geq 0} \left|\omega_{\lambda XY}(\tau_\lambda^t(\mathfrak{F}_j))\right| < \infty.$$

This result and the multi-variable Vitali theorem yield Theorem 1.4 (see Theorem 2.3 in [JOP3]). Moreover, the relations

$$\partial_{X_j}\omega_{\lambda XY+}(\mathfrak{F}_k) = \lim_{t\to+\infty} \partial_{X_j}\omega_{\lambda XY} \circ \tau_\lambda^t(\mathfrak{F}_k), \qquad \partial_{Y_j}\omega_{\lambda XY+}(\mathfrak{F}_k) = \lim_{t\to+\infty} \partial_{Y_j}\omega_{\lambda XY} \circ \tau_\lambda^t(\mathfrak{F}_k), \tag{3.17}$$

hold for $(\lambda, X, Y) \in R_{\Lambda,\delta} \times D_\epsilon$. The proof of Relations (1.9) is completed by invoking the following identities proven in [JOP1, JOP2]:

$$\partial_{X_j}\omega_{\lambda XY}(\tau_\lambda^t(\mathfrak{F}_k))\big|_{X=Y=0} = \frac{1}{\beta_{\rm eq}} \int_0^t {\rm d}s \int_0^{\beta_{\rm eq}} {\rm d}u\, \omega_{\lambda{\rm eq}}(\tau_\lambda^s(\mathfrak{F}_k)\tau_\lambda^{{\rm i}u}(\Phi_j)),$$

$$\partial_{Y_j}\omega_{\lambda XY}(\tau_\lambda^t(\mathfrak{F}_k))\big|_{X=Y=0} = \frac{1}{\beta_{\rm eq}} \int_0^t {\rm d}s \int_0^{\beta_{\rm eq}} {\rm d}u\, \omega_{\lambda{\rm eq}}(\tau_\lambda^s(\mathfrak{F}_k)\tau_\lambda^{{\rm i}u}(\mathcal{J}_j)).$$

Proposition 4.1 in [JOP2] yields that (1.9) and time-reversal invariance (A4) imply (1.10). The KMS condition and (1.10) imply (1.11) [JOP1, JOP2]. Hence, to complete the proofs of Theorems 1.4 and 1.5 we need to establish Theorem 3.1.

3.2 Proof of Theorem 3.1

The GNS representation of the algebra $\mathcal{O}$ associated to the gauge-invariant quasi-free state ω_{XY} can be explicitly computed [AW, BR2]. Let $\mathcal{F}$ be the anti-symmetric Fock space over $\mathfrak{h}$. We denote by $\Omega_{\rm f}$ the vacuum vector and by N the number operator. Let

$$\mathcal{H} = \mathcal{F} \otimes \mathcal{F}, \qquad \Omega = \Omega_{\rm f} \otimes \Omega_{\rm f}.$$

In the sequel $\mathcal{B}(\mathfrak{H})$ denotes the C^*-algebra of all bounded operators on a Hilbert space $\mathfrak{H}$. Let $\mathcal{C}_j$ be given complex conjugations on $\mathfrak{h}_j$ and $\mathcal{C} = \oplus_j \mathcal{C}_j$. Without loss of generality we may assume that $\mathcal{C}_j$ commutes with h_j. As usual, we denote $\mathcal{C}f = \overline{f}$. The map

$$\pi_{XY}(a(f)) = a((I - T_{XY})^{1/2} f) \otimes I + (-I)^N \otimes a^*(T_{XY}^{1/2}\overline{f}),$$

uniquely extends to a representation $\pi_{XY} : \mathcal{O} \to \mathcal{B}(\mathcal{H})$ and the triple $(\mathcal{H}, \pi_{XY}, \Omega)$ is the GNS-representation of the algebra $\mathcal{O}$ associated to the state ω_{XY}.

In what follows we suppose that Assumptions (A1)-(A3) hold. By adding a constant to $\mu_{\rm eq}$ without loss of generality we may assume that $h_j \geq 0$.

Lemma 3.2 *For $\beta > 0$ and $\mu \in \mathbb{R}$ set*

$$\epsilon(\beta,\mu) = \frac{\pi\beta}{\pi + 4\beta(|\mu|+1)} < \beta.$$

The functions

$$l_\pm(s,x,y) = \left(1 + \mathrm{e}^{\pm[(\beta-x)s-(\beta\mu+y)]}\right)^{-1/2},$$

are continuous and, for fixed s, analytic in (x,y) on the set $\{(s,x,y) \in \mathbb{R}_+ \times \mathbb{C}^2 \,|\, |x| < \epsilon(\beta,\mu), |y| < \epsilon(\beta,\mu)\}$. Moreover, for any $\delta < \epsilon(\beta,\mu)$ one has

$$\sup_{s\in\mathbb{R}_+,(x,y)\in\mathbb{C}^2,|x|<\delta,|y|<\delta} |l_\pm(s,x,y)| < \infty.$$

Proof. Set $x = a + \mathrm{i}b$ and $y = c + \mathrm{i}d$ with $a,b,c,d \in \mathbb{R}$, $M_\delta = \{(x,y) \in \mathbb{C}^2 \,|\, |x| < \delta, |y| < \delta\}$ and write the exponent in l_- as

$$\theta(s,x,y) = -\left(u(s)\left(1 - \mathrm{i}\frac{b}{\beta-a}\right) - \mathrm{i}\left(d + b\frac{\beta\mu+c}{\beta-a}\right)\right),$$

where

$$u(s) = (\beta - a)\left(s - \frac{\beta\mu+c}{\beta-a}\right).$$

If $(x,y) \in M_\delta$ with $\delta < \beta$, then

$$\left|\frac{b}{\beta-a}\right| < \frac{\delta}{\beta-\delta}, \qquad \left|d + b\frac{\beta\mu+c}{\beta-a}\right| < \delta\,\frac{\beta(1+|\mu|)}{\beta-\delta},$$

and it follows that $\theta(\mathbb{R}_+ \times M_\delta)$ is contained in the dashed region of Figure 3. An elementary calculation shows that for $\delta < \epsilon(\beta,\mu)$ this region does not intersect the half-lines $\mathbb{R}_+ \pm \mathrm{i}\pi/2$. Another elementary calculation shows that $1 + \mathrm{e}^{\theta(\mathbb{R}_+\times M_\delta)}$ is contained in a bounded region of the half-plane

$$\left\{z \in \mathbb{C} \,\middle|\, \mathrm{Re}\, z > 1 - \mathrm{e}^{-\pi\beta(\delta^{-1}-\epsilon(\beta,\mu)^{-1})}\right\}.$$

Thus, l_- is a bounded continuous function on $\mathbb{R}_+ \times M_\delta$ which is clearly analytic in (x,y) for any fixed $s \in \mathbb{R}_+$. This yields the result since $\mathrm{e}^{\theta/2}$ has obviously the same properties and $l_+ = \mathrm{e}^{\theta/2} l_-$ for real s,x,y. □

The spectral theorem and Lemma 3.2 yield

Lemma 3.3 *The maps*

$$(X,Y) \mapsto (I - T_{XY})^{1/2} \in \mathcal{B}(\mathfrak{h}), \qquad (X,Y) \mapsto T_{XY}^{1/2} \in \mathcal{B}(\mathfrak{h}),$$

extend to analytic $\mathcal{B}(\mathfrak{h})$-valued functions on $D_{\epsilon(\beta_{\mathrm{eq}},\mu_{\mathrm{eq}})}$.

Since for X,Y real, $\|\pi_{XY}(a^\#(f))\| = \|f\|$, Lemma 3.3 implies

Lemma 3.4 *For any $\delta > 0$ there exists $\epsilon(\delta) > 0$ such that for all $f \in \mathfrak{h}$ the operator-valued function*

$$(X,Y) \mapsto \pi_{XY}(a^\#(f)) \in \mathcal{B}(\mathcal{H}),$$

has an analytic extension to $D_{\epsilon(\delta)}$ which satisfies

$$\sup_{(X,Y)\in D_{\epsilon(\delta)}} \|\pi_{XY}(a^\#(f))\| \le (1+\delta)\|f\|.$$

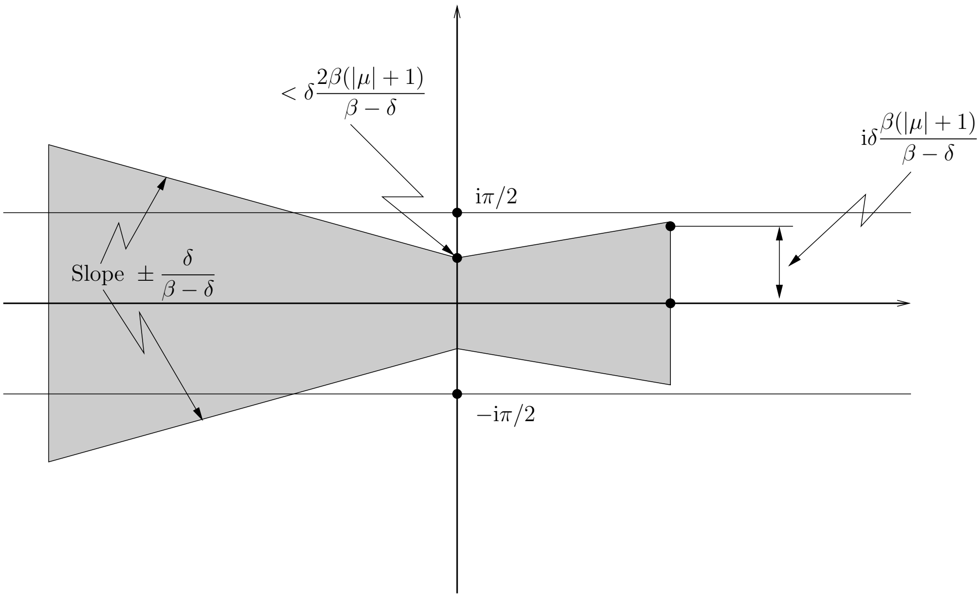

Figure 3: The range of the exponent $\theta(s,x,y)$.

Recall that δ_{XY} is defined by (3.16). Let

$$h_{XY} = h_0 - \mu_{\rm eq} - \sum_j \frac{X_j}{\beta_{\rm eq}} h_j - \sum_j \frac{Y_j}{\beta_{\rm eq}} p_j$$

$$= \sum_j \left[\frac{\beta_{\rm eq} - X_j}{\beta_{\rm eq}} h_j - \frac{\beta_{\rm eq}\mu_{\rm eq} + Y_j}{\beta_{\rm eq}} p_j \right],$$

where p_j is the orthogonal projection on $\mathfrak{h}_j$. Clearly, $\mathrm{e}^{t\delta_{XY}}(a^\#(f)) = a^\#(\mathrm{e}^{\mathrm{i}th_{XY}} f)$ is, for fixed t, an analytic function of X, Y.

Set

$$V_{XY}(s) = \sum_{k=1}^{K} \prod_{j=1}^{n_k} a^*(\mathrm{e}^{-sh_{XY}} u_{kj}) a(\mathrm{e}^{sh_{XY}} v_{kj}),$$

and

$$\mathcal{G}_{\lambda XY} = \mathbb{1} + \sum_{n \geq 1} (-\lambda \beta_{\rm eq})^n \int\limits_{0 \leq s_n \leq \cdots \leq s_1 \leq 1} V_{XY}(\beta_{\rm eq} s_n) \cdots V_{XY}(\beta_{\rm eq} s_1)\, \mathrm{d}s_1 \cdots \mathrm{d}s_n.$$

Araki's perturbation theory [Ar, BR2, DJP] yields that for X, Y real the state $\omega_{\lambda XY}$ can be expressed in terms of ω_{XY} as

$$\omega_{\lambda XY}(A) = \frac{\omega_{XY}(A \mathcal{G}_{\lambda XY})}{\omega_{XY}(\mathcal{G}_{\lambda XY})}. \tag{3.18}$$

Lemma 3.5 *The function*

$$(t, \lambda, X, Y) \mapsto \pi_{XY}(\tau_0^t(\mathcal{G}_{\lambda XY})) \in \mathcal{B}(\mathcal{H}),$$

extends to a continuous function on $\mathbb{R} \times \mathbb{C} \times D_{\epsilon(\beta_{\rm eq}, \mu_{\rm eq})}$ *which is analytic in* (λ, X, Y) *for fixed* t*. Moreover, for all* $\Lambda > 0$ *and* $0 < \epsilon < \epsilon(\beta_{\rm eq}, \mu_{\rm eq})$*,*

$$\sup_{t \in \mathbb{R}, \lambda \in \mathbb{C}, |\lambda| < \Lambda, (X,Y) \in D_\epsilon} \|\pi_{XY}(\tau_0^t(\mathcal{G}_{\lambda XY}))\| < \infty.$$

Proof. Since for X, Y real,

$$\pi_{XY}(\tau_0^t(\mathcal{G}_{\lambda XY})) = \mathbb{1} + \sum_{n\geq 1}(-\lambda\beta_{\rm eq})^n \int\limits_{0\leq s_n\leq\cdots\leq s_1\leq 1} \pi_{XY}(V_{XY}(\beta_{\rm eq}s_n, t))\cdots\pi_{XY}(V_{XY}(\beta_{\rm eq}s_1, t))\,{\rm d}s_1\cdots{\rm d}s_n,$$

where

$$\pi_{XY}(V_{XY}(\beta_{\rm eq}s, t)) = \sum_{k=1}^{K}\prod_{j=1}^{n_k}\pi_{XY}(a^*({\rm e}^{-\beta_{\rm eq}sh_{XY}}{\rm e}^{{\rm i}th_0}u_{kj}))\pi_{XY}(a({\rm e}^{\beta_{\rm eq}sh_{XY}}{\rm e}^{{\rm i}th_0}v_{kj})),$$

the statement follows from Lemma 3.4. □

Lemma 3.6 *For all t and $A \in \mathcal{O}$,*

$$\omega_{\lambda XY}(\tau_\lambda^t(A)) = \omega_{\lambda XY}(\tau_0^t(A)) + \frac{{\rm i}\lambda}{\omega_{XY}(\mathcal{G}_{\lambda XY})}\int_0^t \omega_{XY}([V, \tau_\lambda^s(A)]\tau_0^{s-t}(\mathcal{G}_{\lambda XY}))\,{\rm d}s. \tag{3.19}$$

Proof. Relation (3.18) yields

$$(\omega_{\lambda XY}(\tau_\lambda^t(A)) - \omega_{\lambda XY}(\tau_0^t(A)))\omega_{XY}(\mathcal{G}_{\lambda XY}) = \omega_{XY}((\tau_\lambda^t(A) - \tau_0^t(A))\mathcal{G}_{\lambda XY}).$$

Since ω_{XY} is τ_0-invariant we have

$$\begin{aligned}\omega_{XY}((\tau_\lambda^t(A) - \tau_0^t(A))\mathcal{G}_{\lambda XY}) &= \omega_{XY}((\tau_0^{-t}\circ\tau_\lambda^t(A) - A)\tau_0^{-t}(\mathcal{G}_{\lambda XY}))\\ &= {\rm i}\lambda\int_0^t \omega_{XY}(\tau_0^{-s}([V, \tau_\lambda^s(A)])\tau_0^{-t}(\mathcal{G}_{\lambda XY})\,{\rm d}s\\ &= {\rm i}\lambda\int_0^t \omega_{XY}([V, \tau_\lambda^s(A)]\tau_0^{s-t}(\mathcal{G}_{\lambda XY}))\,{\rm d}s,\end{aligned}$$

and (3.19) follows. □

Lemma 3.7 *For any $\Lambda > 0$ there exist $\epsilon > 0$ and $\delta > 0$ such that the function*

$$(\lambda, X, Y) \mapsto \omega_{XY}(\mathcal{G}_{\lambda XY}),$$

extends to an analytic function on $\mathbb{C}\times D_\epsilon$ which satisfies

$$\inf_{\lambda\in R_{\Lambda,\delta},(X,Y)\in D_\epsilon} |\omega_{XY}(\mathcal{G}_{\lambda XY})| > 0. \tag{3.20}$$

Proof. Since $\omega_{XY}(\mathcal{G}_{\lambda XY}) = (\Omega, \pi_{XY}(\mathcal{G}_{\lambda XY})\Omega)$, the first statement is a special case of Lemma 3.5. Since $\omega_{XY}(\mathcal{G}_{\lambda XY}) > 0$ for λ, X, Y real, by continuity (3.20) holds for ϵ and δ small enough. □

Lemma 3.8 *For any $\Lambda > 0$ there exist $\epsilon > 0$ and $\delta > 0$ such that for all $t \in \mathbb{R}$ the functions*

$$(\lambda, X, Y) \mapsto \omega_{\lambda XY}(\tau_0^t(\mathfrak{F}_j)), \tag{3.21}$$

extend to analytic functions on $R_{\Lambda,\delta}\times D_\epsilon$ such that

$$\sup_{\lambda\in R_{\Lambda,\delta},(X,Y)\in D_\epsilon,t\in\mathbb{R}} |\omega_{\lambda XY}(\tau_0^t(\mathfrak{F}_j))| < \infty.$$

Proof. For X, Y real,

$$\omega_{\lambda XY}(\tau_0^t(\mathfrak{F}_j)) = \frac{(\Omega, \pi_{XY}(\mathfrak{F}_j)\pi_{XY}(\tau_0^{-t}(\mathcal{G}_{\lambda XY}))\Omega)}{\omega_{XY}(\mathcal{G}_{\lambda XY})}.$$

This identity and Lemmas 3.3, 3.5, and 3.7 yield the statement. □

Lemma 3.9 *Let $0 < \Lambda < \Lambda_0$ be given. Then there exists $\epsilon > 0$ such that for all $A = a^{\#}(f_1)\cdots a^{\#}(f_m)$ with $f_j \in \mathcal{D}$, the map*

$$(t, \lambda, X, Y) \mapsto \pi_{XY}([V, \tau_\lambda^t(A)]) \in \mathcal{B}(\mathcal{H}), \tag{3.22}$$

extends to a continuous function on $\mathbb{R}_+ \times \{\lambda \in \mathbb{C} \,|\, |\lambda| < \Lambda\} \times D_\epsilon$ which is analytic in (λ, X, Y) for fixed $t \in \mathbb{R}$. Moreover,

$$\int_0^\infty \sup_{\lambda\in\mathbb{C}, |\lambda|<\Lambda, (X,Y)\in D_\epsilon} \|\pi_{XY}([V, \tau_\lambda^t(A)])\| \,\mathrm{d}t < \infty. \tag{3.23}$$

Proof. The expansion (1.1) yields that

$$\pi_{XY}([V, \tau_\lambda^t(A)]) = \pi_{XY}([V, \tau_0^t(A)])$$

$$+ \sum_{n=1}^\infty (\mathrm{i}\lambda)^n \int\limits_{0\le s_n\le\cdots\le s_1\le t} \pi_{XY}([V, [\tau_0^{s_n}(V), [\cdots, [\tau_0^{s_1}(V), \tau_0^t(A)]\cdots]]])\,\mathrm{d}s_1\cdots\mathrm{d}s_n.$$

Set

$$C_{XY}^{(0)} = \pi_{XY}([V, \tau_0^t(A)]),$$

and, for $n \ge 1$,

$$C_{XY}^{(n)}(t, s_1, \ldots, s_n) = \pi_{XY}\left([V, [\tau_0^{s_n}(V), [\cdots, [\tau_0^{s_1}(V), \tau_0^t(A)]\ldots]]]\right).$$

Theorem 1.1 yields that for each n there exist a finite index set $\mathcal{P}_n(A)$, scalar functions $G_{A,p}^{(n)}$ which do not depend on X, Y, and monomials $F_{A,p}^{(n)} \in \mathcal{O}$ such that

$$C_{XY}^{(n)}(t, s_1, \ldots, s_n) = \sum_{p\in\mathcal{P}_n(A)} G_{A,p}^{(n)}(t, s_1, \ldots, s_n)\pi_{XY}(F_{A,p}^{(n)}).$$

Recall our standing assumption (1.3) and Part (2) of Theorem 1.1. Let $\delta > 0$ be such that

$$C_0 \equiv (1+\delta)^{2\overline{n}-1} < \frac{\Lambda_0}{\Lambda}. \tag{3.24}$$

Applying Lemma 3.4 with this δ to the factors of $\pi_{XY}(F_{A,p}^{(n)})$ we conclude that there exists $\epsilon > 0$ (which depends on δ) such that for all n the functions

$$(X, Y) \mapsto \pi_{XY}(F_p^{(n)}) \in \mathcal{B}(\mathcal{H}),$$

extend to analytic functions on D_ϵ satisfying

$$\sup_{t, s_1, \ldots s_n\in\mathbb{R}, (X,Y)\in D_\epsilon} \|\pi_{XY}(F_p^{(n)})\| \le C_1 C_0^{n+1},$$

where $C_1 = (1+\delta)^{m-1}[\max(1, \|f_1\|, \cdots, \|f_m\|)]^{m-1}$. By Part (3) of Theorem 1.1,

$$\sum_{n=0}^\infty |\Lambda|^{n+1} \mathrm{C}_0^{n+1} \sum_{p\in\mathcal{P}_n(A)} \int\limits_{0\le s_n\le\cdots\le s_1\le t<\infty} |G_{A,p}^{(n)}(t, s_1, \ldots, s_n)|\,\mathrm{d}t\mathrm{d}s_1\cdots\mathrm{d}s_n < \infty,$$

and we conclude that
$$\int_0^\infty \sup_{\lambda\in\mathbb{C},|\lambda|<\Lambda,(X,Y)\in D_\epsilon} \|\pi_{XY}([V,\tau_\lambda^t(\mathfrak{F}_j)])\| \,\mathrm{d}t < \infty.$$
□

We are now ready to complete:

Proof of Theorem 3.1. We start with formula (3.19). By Lemmas 3.7 and 3.8, it suffices to show that for some $\epsilon > 0$ the functions
$$(\lambda, X, Y) \mapsto \int_0^t (\Omega, \pi_{XY}([V,\tau_\lambda^s(\mathfrak{F}_j)])\pi_{XY}(\tau_0^{s-t}(\mathcal{G}_{\lambda XY}))\Omega)\,\mathrm{d}s,$$
extend to analytic functions on $\{\lambda\in\mathbb{C}\,|\,|\lambda|<\Lambda\}\times D_\epsilon$ such that
$$\sup_{\lambda\in\mathbb{C},|\lambda|<\Lambda,(X,Y)\in D_\epsilon,t\geq 0} \left|\int_0^t (\Omega, \pi_{XY}([V,\tau_\lambda^s(\mathfrak{F}_j)])\pi_{XY}(\tau_0^{s-t}(\mathcal{G}_{\lambda XY}))\Omega)\,\mathrm{d}s\right| < \infty.$$
By Lemma 3.5, it suffices to show that the functions
$$(t,\lambda,X,Y)\mapsto \pi_{XY}([V,\tau_\lambda^t(\mathfrak{F}_j)])\in\mathcal{B}(\mathcal{H}),$$
extend to continuous functions on $\mathbb{R}_+\times\{\lambda\in\mathbb{C}\,|\,|\lambda|<\Lambda\}\times D_\epsilon$ which, for fixed t, are analytic in (λ,X,Y) and satisfy the bound
$$\int_0^\infty \sup_{\lambda\in\mathbb{C},|\lambda|<\Lambda,(X,Y)\in D_\epsilon} \|\pi_{XY}([V,\tau_\lambda^t(\mathfrak{F}_j)])\| \,\mathrm{d}t < \infty.$$
By (A2) and (A3), every $\mathfrak{F}_j$ can be written as a finite sum of monomials $a^\#(f_1)\cdots a^\#(f_m)$ with $f_k\in\mathcal{D}$, and the result follows from Lemma 3.9. □

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